Correction du devoir Maison

Exercice 1 .

Calculons les limites :

♣ $\lim\limits_{x\longrightarrow 3}\dfrac{x^2-x-6}{x^2-4x+3}=\text{"}\dfrac{0}{0}\text{"}\quad (F.I)$

On a $x^2-x-6=(x+2)(x-3)$ *et* $x^2-4x+3=(x-1)(x-3)$ *donc on obtient*

$$\lim_{x\longrightarrow 3}\frac{x^2-x-6}{x^2-4x+3}=\lim_{x\longrightarrow 3}\frac{(x+2)(x-3)}{(x-1)(x-3)}=\lim_{x\longrightarrow 3}\frac{x+2}{x-1}=\frac{5}{2}.$$

♣ $\lim\limits_{x\longrightarrow 2}\dfrac{\sqrt{x+2}+\sqrt{7+x}-5}{x^2-3x+2}=\text{"}\dfrac{0}{0}\text{"}\quad (F.I)$

On a $x^2-3x+2=(x-1)(x-2)$ *donc*

$$\begin{aligned}
\lim_{x\longrightarrow 2}\frac{\sqrt{x+2}+\sqrt{7+x}-5}{x^2-3x+2} &= \lim_{x\longrightarrow 2}\frac{\sqrt{x+2}-2+\sqrt{7+x}-3}{(x-1)(x-2)}\\
&= \lim_{x\longrightarrow 2}\frac{\sqrt{x+2}-2}{(x-1)(x-2)}+\frac{\sqrt{7+x}-3}{(x-1)(x-2)}\\
&= \lim_{x\longrightarrow 2}\frac{(x+2)-4}{(x-1)(x-2)\left(\sqrt{x+2}+2\right)}+\frac{(7+x)-9}{(x-1)(x-2)\left(\sqrt{7+x}+3\right)}\\
&= \lim_{x\longrightarrow 2}\frac{x-2}{(x-1)(x-2)\left(\sqrt{x+2}+2\right)}+\frac{x-2}{(x-1)(x-2)\left(\sqrt{7+x}+3\right)}\\
&= \lim_{x\longrightarrow 2}\frac{1}{(x-1)\left(\sqrt{x+2}+2\right)}+\frac{1}{(x-1)\left(\sqrt{7+x}+3\right)}\\
&= \frac{1}{(2-1)\left(\sqrt{2+2}+2\right)}+\frac{1}{(2-1)\left(\sqrt{7+2}+3\right)}\\
&= \frac{5}{12}
\end{aligned}$$

donc $\lim\limits_{x\longrightarrow 2}\dfrac{\sqrt{x+2}+\sqrt{7+x}-5}{x^2-3x+2}=\dfrac{5}{12}.$

♣ $\lim\limits_{x\longrightarrow -\infty}\sqrt{x^2+x-1}+x = " +\infty-\infty" \ (F.I)$

$$\begin{aligned}
\lim_{x\longrightarrow -\infty}\sqrt{x^2+x-1}+x &= \lim_{x\longrightarrow -\infty}\frac{\left(\sqrt{x^2+x-1}+x\right)\left(\sqrt{x^2+x-1}-x\right)}{\sqrt{x^2+x-1}-x}\\
&= \lim_{x\longrightarrow -\infty}\frac{x^2+x-1-x^2}{\sqrt{x^2+x-1}-x}\\
&= \lim_{x\longrightarrow -\infty}\frac{x-1}{\sqrt{x^2+x-1}-x}\\
&= \lim_{x\longrightarrow -\infty}\frac{x\left(1-\frac{1}{x}\right)}{\sqrt{x^2\left(1+\frac{1}{x}-\frac{1}{x}\right)}-x}\\
&= \lim_{x\longrightarrow -\infty}\frac{x\left(1-\frac{1}{x}\right)}{-x\sqrt{1+\frac{1}{x}-\frac{1}{x}}-x}\\
&= \lim_{x\longrightarrow -\infty}\frac{x\left(1-\frac{1}{x}\right)}{-x\left(\sqrt{1+\frac{1}{x}-\frac{1}{x}}+1\right)}\\
&= \lim_{x\longrightarrow -\infty}-\frac{1-\frac{1}{x}}{\sqrt{1+\frac{1}{x}-\frac{1}{x}}+1} = -\frac{1}{2}
\end{aligned}$$

$donc \lim\limits_{x\longrightarrow -\infty}\sqrt{x^2+x-1}+x = -\frac{1}{2}.$

♣ $\lim\limits_{x\longrightarrow +\infty}\sqrt{2x-1}-3x^2+x+2 = " +\infty-\infty" \ (F.I)$

$$\begin{aligned}
\lim_{x\longrightarrow +\infty}\sqrt{2x-1}-3x^2+x+2 &= \lim_{x\longrightarrow +\infty}\sqrt{x^2\left(\frac{2}{x}-\frac{1}{x^2}\right)}-3x^2+x+2\\
&= \lim_{x\longrightarrow +\infty}x\sqrt{\frac{2}{x}-\frac{1}{x^2}}+x\left(-3x+1+\frac{2}{x}\right)\\
&= \lim_{x\longrightarrow +\infty}x\left(\sqrt{\frac{2}{x}-\frac{1}{x^2}}+\left(-3x+1+\frac{2}{x}\right)\right)\\
&= -\infty
\end{aligned}$$

$$car \lim_{x\longrightarrow +\infty}-3x+1+\frac{2}{x} = -\infty \quad et \quad \lim_{x\longrightarrow +\infty}\sqrt{\frac{2}{x}-\frac{1}{x^2}} = 0$$

Exercice 2 .

Calculons les limites :

♣ $\lim\limits_{x\longrightarrow 0}\dfrac{\sin(2x)-2\sin x}{x^2} = "\dfrac{0}{0}" \quad (F.I)$

Soit $x\in\mathbb{R},$ *on a* $\sin(2x)-2\sin x = 2\sin x.\cos x - 2\sin x = 2\sin x(\cos x - 1),$ *donc*

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{\sin(2x)-2\sin x}{x^2} &= \lim_{x\longrightarrow 0}\frac{2\sin x(\cos x-1)}{x^2}\\ &= \lim_{x\longrightarrow 0} -2\sin x\times\frac{(1-\cos x)}{x^2} = 0\times\frac{1}{2} = 0\end{aligned}$$

♣ $\lim\limits_{x\longrightarrow\frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{\sqrt{2}\sin x - 1} = \text{"}\dfrac{0}{0}\text{"} \quad (F.I)$

$$
\begin{aligned}
\lim_{x\longrightarrow\frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\sin x - 1} &= \lim_{x\longrightarrow\frac{\pi}{4}} \frac{\sqrt{2}\left(\cos x - \dfrac{1}{\sqrt{2}}\right)}{\sqrt{2}\left(\sin x - \dfrac{1}{\sqrt{2}}\right)} \\
&= \lim_{x\longrightarrow\frac{\pi}{4}} \frac{\cos x - \dfrac{\sqrt{2}}{2}}{\sin x - \dfrac{\sqrt{2}}{2}} \\
&= \lim_{x\longrightarrow\frac{\pi}{4}} \frac{\cos x - \cos\dfrac{\pi}{4}}{\sin x - \sin\dfrac{\pi}{4}} \\
&= \lim_{x\longrightarrow\frac{\pi}{4}} \frac{-2\sin\left(\dfrac{x+\frac{\pi}{4}}{2}\right).\sin\left(\dfrac{x-\frac{\pi}{4}}{2}\right)}{2\sin\left(\dfrac{x-\frac{\pi}{4}}{2}\right).\cos\left(\dfrac{x+\frac{\pi}{4}}{2}\right)} \\
&= \lim_{x\longrightarrow\frac{\pi}{4}} -\frac{\sin\left(\dfrac{x+\frac{\pi}{4}}{2}\right)}{\cos\left(\dfrac{x+\frac{\pi}{4}}{2}\right)} \\
&= \lim_{x\longrightarrow\frac{\pi}{4}} -\tan\left(\dfrac{x+\frac{\pi}{4}}{2}\right) \\
&= \lim_{x\longrightarrow\frac{\pi}{4}} -\tan\left(\frac{x}{2}+\frac{\pi}{8}\right) = -\tan\left(\frac{\pi}{8}+\frac{\pi}{8}\right) = -1
\end{aligned}
$$

donc $\lim\limits_{x\longrightarrow\frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{\sqrt{2}\sin x - 1} = -1.$

♣ $\lim\limits_{x\longrightarrow 0^+} \dfrac{1-\cos\sqrt{x}}{\sin x} = \text{"}\dfrac{0}{0}\text{"} \quad (F.I)$

On a

$$\begin{aligned}
\lim_{x\longrightarrow 0^+} \frac{1-\cos\sqrt{x}}{\sin x} &= \lim_{x\longrightarrow 0^+} \left(1-\cos\sqrt{x}\right)\times\frac{1}{\sin x}\\
&= \lim_{x\longrightarrow 0^+} \left(\frac{1-\cos\sqrt{x}}{x}\right)\times\frac{x}{\sin x}\\
&= \lim_{x\longrightarrow 0^+} \left(\frac{1-\cos\sqrt{x}}{\sqrt{x^2}}\right)\times\frac{x}{\sin x}\\
&= \lim_{x\longrightarrow 0^+} \left(\frac{1-\cos\sqrt{x}}{\sqrt{x^2}}\right)\times\frac{1}{\frac{\sin x}{x}}\\
&= \frac{1}{2}\times 1 = \frac{1}{2}
\end{aligned}$$

donc $\lim_{x\longrightarrow 0^+} \frac{1-\cos\sqrt{x}}{\sin x} = \frac{1}{2}$.

Exercice 3 .

On considère la fonction f définie par

$$\begin{cases} f(x) = \dfrac{\sqrt{x}-1}{2-\sqrt{3+x}} & si \quad x>1 \\ \\ f(x) = \dfrac{\sqrt{1-x}}{2x^2+x-3} & si \quad x<1 \end{cases}$$

1. *Montrons que* $D_f = \left]-\infty, \frac{-3}{2}\right[\cup \left]\frac{-3}{2}, 1\right[\cup]1,+\infty[$:

   On pose :

$$f_1(x) = \frac{\sqrt{x}-1}{2-\sqrt{3+x}} \quad et \quad I =]1,+\infty[\quad , \quad f_2(x) = \frac{\sqrt{1-x}}{2x^2+x-3} \quad et \quad J =]-\infty, 1[$$

On a d'une part

$$\begin{aligned}
D_{f_1} &= \left\{x\in\mathbb{R}/\ x\geq 0\ \ et\ 3+x\geq 0\ \ et\ 2-\sqrt{3+x}\neq 0\right\}\\
&= \left\{x\in\mathbb{R}/\ x\geq 0\ \ et\ x\geq -3\ \ et\ \sqrt{3+x}\neq 2\right\}\\
&= \left\{x\in\mathbb{R}/\ x\geq 0\ \ et\ \sqrt{3+x}\neq 2\right\}
\end{aligned}$$

soit $x\in[0,+\infty[$, *on résout l'équation :* $\sqrt{3+x}=2$

$$\sqrt{3+x} = 2 \iff 3+x=4 \iff x=4-3 \iff x=1$$

donc :

$$\begin{aligned}
D_{f_1} &= \{x\in\mathbb{R}/\ x\geq 0\ \ et\ x\neq 1\}\\
&= [0,1[\cup]1,+\infty[
\end{aligned}$$

par suite :

$$\begin{aligned} D_{f_1} \cap I &= ([0,1[\cup]1,+\infty[) \cap]1,+\infty[\\ &= ([0,1[\cap]1,+\infty[) \cup (]1,+\infty[\cap]1,+\infty[) \\ &= \varnothing \cup]1,+\infty[\\ &=]1,+\infty[\end{aligned}$$

D'autre part

$$\begin{aligned} D_{f_2} &= \left\{x \in \mathbb{R}/\ 1-x \geq 0\ \ et\ 2x^2+x-3 \neq 0\right\} \\ &= \left\{x \in \mathbb{R}/\ x \leq 1\ \ et\ 2x^2+x-3 \neq 0\right\} \end{aligned}$$

Résolvons l'équation : $2x^2+x-3=0$.

Calculons Δ.

$$\Delta = b^2-4ac = 1-4\times 2\times(-3) = 25 > 0$$

donc l'équation admet deux solutions réelles distinctes x_1 *et* x_2 :

$$x_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-1+\sqrt{25}}{4} = \frac{4}{4} = 1 \quad et \quad x_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-1-\sqrt{25}}{4} = \frac{-6}{4} = \frac{-3}{2}$$

d'où

$$\begin{aligned} D_{f_2} &= \left\{x \in \mathbb{R}/\ x \leq 1\ \ et\ x \neq 1\ \ et\ x \neq \frac{-3}{2}\right\} \\ &= \left]-\infty, \frac{-3}{2}\right[\cup \left]\frac{-3}{2}, 1\right[\end{aligned}$$

par suite :

$$\begin{aligned} D_{f_2} \cap J &= \left(\left]-\infty, \frac{-3}{2}\right[\cup \left]\frac{-3}{2}, 1\right[\right) \cap]-\infty, 1[\\ &= \left(\left]-\infty, \frac{-3}{2}\right[\cap]-\infty,1[\right) \cup \left(\left]\frac{-3}{2}, 1\right[\cap]-\infty,1[\right) \\ &= \left]-\infty, \frac{-3}{2}\right[\cup \left]\frac{-3}{2}, 1\right[\end{aligned}$$

Finalement :

$$\begin{aligned} D_f &= (D_{f_1} \cap I) \cup (D_{f_2} \cap J) \\ &= \left]-\infty, \frac{-3}{2}\right[\cup \left]\frac{-3}{2}, 1\right[\cup]1,+\infty[\end{aligned}$$

2. *Calculons :* $\lim\limits_{x\longrightarrow+\infty} f(x)$ *et* $\lim\limits_{x\longrightarrow-\infty} f(x)$.

- $$\begin{aligned}
\lim_{x\longrightarrow+\infty} f(x) &= \lim_{x\longrightarrow+\infty} \frac{\sqrt{x}-1}{2-\sqrt{3+x}} \\
&= \lim_{x\longrightarrow+\infty} \frac{\sqrt{x}-1}{2-\sqrt{x}\times\sqrt{1+\frac{3}{x}}} \\
&= \lim_{x\longrightarrow+\infty} \frac{\sqrt{x}\left(1-\frac{1}{\sqrt{x}}\right)}{\sqrt{x}\left(\frac{2}{\sqrt{x}}-\sqrt{1+\frac{3}{x}}\right)} \\
&= \lim_{x\longrightarrow+\infty} \frac{1-\frac{1}{\sqrt{x}}}{\frac{2}{\sqrt{x}}-\sqrt{1+\frac{3}{x}}} = -1
\end{aligned}$$

- $$\begin{aligned}
\lim_{x\longrightarrow-\infty} f(x) &= \lim_{x\longrightarrow-\infty} \frac{\sqrt{1-x}}{2x^2+x-3} \\
&= \lim_{x\longrightarrow-\infty} \frac{(1-x)}{(2x^2+x-3)\sqrt{1-x}} \\
&= \lim_{x\longrightarrow-\infty} \frac{1-x}{2x^2+x-3} \times \frac{1}{\sqrt{1-x}} = 0 \\
car \quad &: \quad \lim_{x\longrightarrow-\infty} \frac{1-x}{2x^2+x-3} = \lim_{x\longrightarrow-\infty} \frac{-x}{2x^2} = \lim_{x\longrightarrow-\infty} \frac{-1}{2x} = 0 \quad et \quad \lim_{x\longrightarrow-\infty} \frac{1}{\sqrt{1-x}} = 0
\end{aligned}$$

3. *Calculons* : $\lim_{x\longrightarrow1^+} f(x)$ *et* $\lim_{x\longrightarrow1^-} f(x)$.

- $\lim_{x\longrightarrow1^+} f(x) = \lim_{x\longrightarrow1^+} \frac{\sqrt{x}-1}{2-\sqrt{3+x}} = "\frac{0}{0}"$ $(F.I)$

$$\begin{aligned}
\lim_{x\longrightarrow1^+} f(x) &= \lim_{x\longrightarrow1^+} \frac{\sqrt{x}-1}{2-\sqrt{3+x}} \\
&= \lim_{x\longrightarrow1^+} \frac{(\sqrt{x}-1)(\sqrt{x}+1)\left(2+\sqrt{3+x}\right)}{\left(2-\sqrt{3+x}\right)\left(2+\sqrt{3+x}\right)(\sqrt{x}+1)} \\
&= \lim_{x\longrightarrow1^+} \frac{(x-1)\left(2+\sqrt{3+x}\right)}{(4-3-x)(\sqrt{x}+1)} \\
&= \lim_{x\longrightarrow1^+} \frac{(x-1)\left(2+\sqrt{3+x}\right)}{(1-x)(\sqrt{x}+1)} \\
&= \lim_{x\longrightarrow1^+} \frac{(x-1)\left(2+\sqrt{3+x}\right)}{-(x-1)(\sqrt{x}+1)} \\
&= \lim_{x\longrightarrow1^+} -\frac{\left(2+\sqrt{3+x}\right)}{(\sqrt{x}+1)} = -2
\end{aligned}$$

- $\lim\limits_{x \longrightarrow 1^-} f(x) = \lim\limits_{x \longrightarrow 1^-} \dfrac{\sqrt{1-x}}{2x^2+x-3} = \text{"}\dfrac{0}{0}\text{"}\ (F.I)$

$$\begin{aligned}
\lim_{x \longrightarrow 1^-} f(x) &= \lim_{x \longrightarrow 1^-} \frac{\sqrt{1-x}}{2x^2+x-3} \\
&= \lim_{x \longrightarrow 1^-} \frac{(1-x)}{(2x+3)(x-1)\sqrt{1-x}} \\
&= \lim_{x \longrightarrow 1^-} \frac{-(x-1)}{(2x+3)(x-1)\sqrt{1-x}} \\
&= \lim_{x \longrightarrow 1^-} \frac{-1}{(2x+3)\sqrt{1-x}} = -\infty \\
car &: \quad \lim_{x \longrightarrow 1^-} \sqrt{1-x} = 0^+
\end{aligned}$$

Donc la fonction n'admet pas une limite en 1.

4. *La limite en* $x_1 = \dfrac{-3}{2}$.

   On a

$$\lim_{x \longrightarrow \frac{-3}{2}} \frac{\sqrt{1-x}}{2x^2+x-3} = \frac{\sqrt{\frac{5}{2}}}{0} = \infty$$

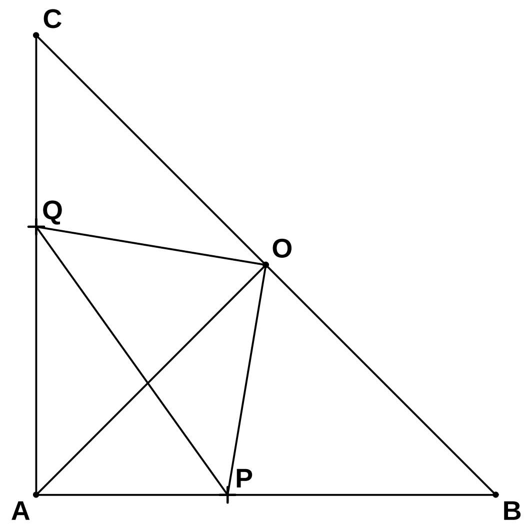
On étudie le signe du trinôme $2x^2+x-3$ *pour tout* x *de* $\mathbb{R}$.

x	$-\infty$		$-3/2$		1		$+\infty$
$2x^2+x-3$		$+$	0	$-$	0	$+$	

Donc :

$$\lim_{x \longrightarrow \frac{-3}{2}^+} \frac{\sqrt{1-x}}{2x^2+x-3} = -\infty \quad et \quad \lim_{x \longrightarrow \frac{-3}{2}^-} \frac{\sqrt{1-x}}{2x^2+x-3} = +\infty$$

Exercice 4 .



1. *On cherche* $r(A)$ *et* $r(C)$:

   On considère la rotation de centre O *et d'angle* $\dfrac{\pi}{2}$.

   ♣ *On a* $OA = OB$ *et* $\left(\overrightarrow{OA}, \overrightarrow{OB}\right) \equiv \dfrac{\pi}{2}\,[2\pi]$ *donc* $r(A) = B$.

   ♣ *On a* $OC = OA$ *et* $\left(\overrightarrow{OC}, \overrightarrow{OA}\right) \equiv \dfrac{\pi}{2}\,[2\pi]$ *donc* $r(C) = A$.

2. *Montrons que :* $r(Q) = P$.

   On pose $r(Q) = Q'$. *On a* $\overrightarrow{CQ} = \dfrac{2}{5}\overrightarrow{CA}$ *comme* $r(A) = B$ *et* $r(C) = A$ *donc* $\overrightarrow{AQ'} = \dfrac{2}{5}\overrightarrow{AB}$ *car la rotation conserve le coefficient de colinéairité de deux vecteurs. Or* $\overrightarrow{AP} = \dfrac{2}{5}\overrightarrow{AB}$ *donc on déduit que*

   $$\overrightarrow{AQ'} = \overrightarrow{AP}$$

   ceci signifie que $Q' = P$. *D'où* $r(Q) = P$.

3. *La nature du triangle* OPQ :

   On a $r(Q) = P$ *c'est équivant à* $OQ = OP$ *et* $\left(\overrightarrow{OQ}, \overrightarrow{OP}\right) \equiv \dfrac{\pi}{2}\,[2\pi]$. *Donc le nature du triangle* OPQ *est rectangle isocèle en* O.

FIN

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